Identical Particles.

Generally, two particle wave function with a particular total energy can be written as
$$\Psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2) e^{-iEt/\hbar}$$

If particle 1 is in the one-particle state $\psi_a(\vec{r})$ and particle 2 is in the state $\psi_b(\vec{r})$,
$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2)$$

However, this is possible only if we can distinguish particle 1 and particle 2.

In quantum mechanics, two identical particles are indistinguishable. Therefore, it is impossible to assign each particle to a particular state.
Instead, there are two valid ways to write the wave function, not assigning each particle to a particular state.
$$\psi_+(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) + \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right]$$
$$\psi_-(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) - \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right]$$

Particles satisfying the first rule are called "boson", and those satisfying the second are called "fermion".

It turns out that

Integer spins $\Rightarrow$ boson
half integer spins $\Rightarrow$ fermion

For boson, $\psi_+(\vec{r}_2, \vec{r}_1) = \psi_+(\vec{r}_1, \vec{r}_2)$

For fermion, $\psi_-(\vec{r}_2, \vec{r}_1) = -\psi_-(\vec{r}_1, \vec{r}_2)$

For the case of fermion, if ψ_a and ψ_b are the same states,

$$\psi_-(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_1)\psi_a(\vec{r}_2)\right] = 0$$

In other words, two identical fermions cannot occupy the same state.

$\Rightarrow$ Called "Pauli exclusion Principle"

$\boxed{\text{Ex}_1}$ Consider two noninteracting particles in the infinite potential well.

As we know, the one particle states are

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right),\ E_n = n^2 K,\ K \equiv \frac{\pi^2\hbar^2}{2ma^2}$$

Ignoring the spins, find the ground state for ① two distinguishable particles
② two identical bosons and
③ two identical fermions.

① $$\psi_{11} = \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right),$$

$$E_G = E_{11} = (1^2 + 1^2) K = 2K$$

② two identical bosons,

$$\psi_{11} = \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right)$$

$$E_G = E_{11} = 2K$$

③ two identical fermions,

$$\psi = \frac{1}{\sqrt{2}}\left(\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2) \right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{2}{a} \left(\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi}{a} x_1\right) \sin\left(\frac{\pi}{a} x_2\right) \right)$$

$$\frac{1}{\sqrt{2}} \cdot \frac{2}{a} = \frac{\sqrt{2}}{a}$$

$$E_G = E_{12} = (1+4) K = 5K$$

* Now consider two identical spin $\frac{1}{2}$ particles. What is the ground state and the energy? Still assume they are not interacting.

The complete state has to be

$$|S\rangle = \psi(x_1, x_2) \underset{\uparrow \text{ spinor}}{\chi(s_1, s_2)}$$

As we have seen in the previous lecture, the total spin of two $\frac{1}{2}$ spins can be

$$\chi_0 = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

the singlet state or

$$\chi_1 = \left\{ \begin{array}{c} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right) \\ |\downarrow\downarrow\rangle \end{array} \right) \text{ the triplet state}$$

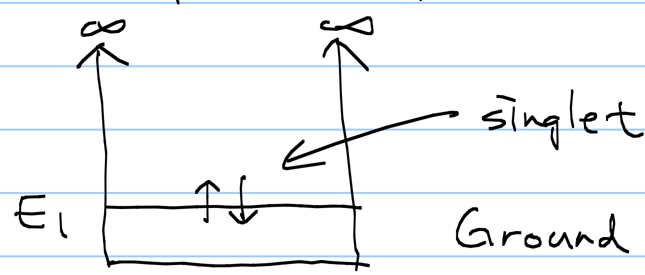
The singlet state is symmetric w.r.t. the exchange of particle 1 and 2, and the triplet state is antisymmetric.

Thus the ground state should be

$$|GS\rangle = \underbrace{\underbrace{\psi_1(x_1)\psi_1(x_2)}_{\text{symmetric}} \underbrace{\chi_0(s_1, s_2)}_{\text{antisymmetric}}}_{\text{antisymmetric}}$$

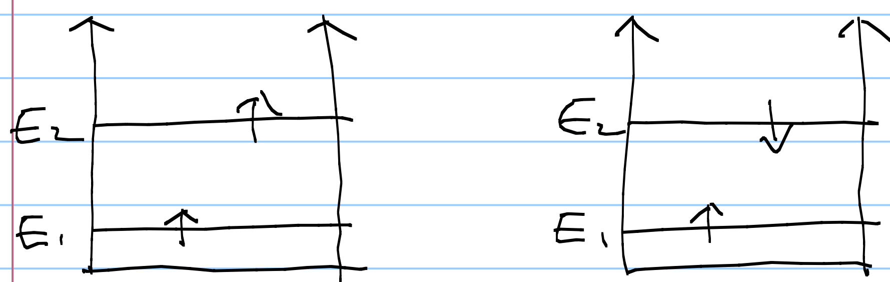
$$= \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right) \chi_0$$

$$E_G = E_{11} = 2K$$



Ground state configuration for two non-interacting spin $\frac{1}{2}$ particles.

Now still neglecting the e-e interaction, what is the first excited state (and its energy) and its degeneracy?



triplet state $\rightarrow$ degeneracy three

singlet state $\rightarrow$ degeneracy one

Total degeneracy of four with the energy of $E_1 + E_2 = 5K$

The triplet state should look like

$$|\psi\rangle_t = \frac{1}{\sqrt{2}}\left\{\psi_{n=1}(x_1)\,\psi_{n=2}(x_2) - \psi_{n=2}(x_1)\,\psi_{n=1}(x_2)\right\} \times \chi_1$$

, where χ_1 is one of $\begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |\downarrow\downarrow\rangle \end{cases}$

So $|\Delta\rangle_t$ can be one of these three states.

Here note that for the triplet state, the spin part is symmetric w.r.t. the interchange of particles, and the spatial part is antisymmetric, the entire state being antisymmetric

Now the singlet state is

$$|\Delta\rangle_s = \psi_{n=1}(x_1)\,\psi_{n=2}(x_2)\,\chi_0$$

, where $\chi_0 = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$.

Again the spin part is antisymmetric w.r.t. the particle interchange. Therefore the spatial part should be symmetric.

So without electron-electron electrons, these four states have the same energy of $E = E_1 + E_2 = 5K$

; K was defined last time as $\frac{\hbar^2 \pi^2}{2ma^2}$